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## On the BRST operator of $W$ -strings

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We discuss the conditions under which the BRST operator of a  $W$ -string can be written as the sum of two operators that are separately nilpotent and anticommute with each other. We illustrate our results with the example of the non-critical  $W_3$ -string. Furthermore, we apply our results to make a conjecture about a relationship between the spectrum of a non-critical  $W_n$ -string and a  $W_{n-1}$ -string.

### 1. Introduction

In order to deal with a system of first-class constraints, it is convenient to use the BRST formalism [1]. In particular, this approach has turned out to be rather fruitful in the study of string theories [2,3] and, more recently, in the study of critical [4–6] and non-critical [7–10]  $W$ -string theories<sup>#1</sup>. In the quantum case, the physical states are defined as the cohomology classes of a nilpotent BRST operator  $Q_B$ . In the case of strings and  $W$ -strings, the determination of these cohomology classes is a rather complicated task and, so far, has only been solved completely for critical [2] and non-critical [3] strings.

Recently, it has been pointed out [4] that, in the case of a critical  $W_3$ -string, the relevant BRST operator can be written as the sum of two operators that are separately nilpotent and anticommute with each other, i.e.

$$Q_B = Q_0 + Q_1, \quad Q_0^2 = Q_1^2 = Q_0 Q_1 + Q_1 Q_0 = 0. \quad (1)$$

One thus ends up with a so-called *double complex* which leads to a double grading  $(m, n)$  such that  $Q_0$  has grading  $(1, 0)$  and  $Q_1$  has grading  $(0, 1)$ . One would expect that, due to this additional structure, the determination of the cohomology of  $Q_B$  simplifies. For instance, each solution of the  $Q_0$  and  $Q_1$  cohomology can now be used to construct a solution of the full  $Q_B$  cohomology. In view of these expected simplifications, it is of interest to see whether the split-up of the BRST operator, as given in (1), also occurs under more general circumstances. It is the purpose of this paper to investigate this issue. Our strategy will be to first discuss the generic situation for classical systems. Next, we will consider the quantisation. We will illustrate our results with the example of the classical  $w_3$ -algebra and the recently introduced modified  $w_3$ -algebra [7,15]. At the end of this letter we will comment on the case of general  $w_n$ -algebras and make a conjecture about the spectrum of the non-critical  $W_n$ -string.

<sup>#1</sup> The possibility of a  $W_3$ -string was first discussed in [11]. The BRST-operator of the  $W_3$  algebra has been constructed in [12], while in [13] the BRST-operator for general quadratically nonlinear algebras was obtained.

## 2. The classical case

In the classical BRST formalism, instead of imposing constraints, one first extends the standard phase space of coordinates and momenta with additional anticommuting ghost coordinates. To be more precise, let us assume that we have coordinates and momenta  $z_A$  with canonical Poisson brackets  $\{z_A, z_B\} = \Omega_{AB}$ . The matrix  $\Omega$  defines a symplectic structure on the phase space. Furthermore, let  $\Phi_\alpha(z_A)$  be a set of first-class constraints that form the following Poisson-bracket algebra

$$\{\Phi_\alpha, \Phi_\beta\} = f_{\alpha\beta}{}^\gamma(z_A) \Phi_\gamma. \quad (2)$$

To each of the constraints we associate a canonical pair of ghosts  $c^\alpha, b_\alpha$  with Poisson bracket  $\{c^\alpha, b_\beta\} = -\delta^\alpha_\beta$ . In the BRST approach one constructs, corresponding to this system, a nilpotent BRST charge  $Q_B$ , i.e.  $\{Q_B, Q_B\} = 0$  (see e.g. [14]). The generic structure of this BRST charge is given by

$$Q_B = c^\alpha \Phi_\alpha - \frac{1}{2} f_{\alpha\beta}{}^\gamma c^\beta c^\alpha b_\gamma + \dots, \quad (3)$$

where the dots indicate terms of higher order in the ghosts. If we assume that the structure constants satisfy the condition

$$f_{[\alpha\beta}{}^\epsilon f_{\gamma]\epsilon}{}^\delta - \{f_{[\alpha\beta}{}^\delta, \Phi_{\gamma]}\} = 0, \quad (4)$$

then the BRST charge only contains terms linear and trilinear in the ghosts. For simplicity, we will restrict our discussion to algebras of this type. Let  $\Phi_\alpha(z_A) = \{T_a(z_A), W_i(z_A)\}$ , then we also assume that in terms of  $T_a$  and  $W_i$  the constraint algebra can be written as

$$\begin{aligned} \{T_a, T_b\} &= f_{ab}{}^c T_c, \\ \{T_a, W_i\} &= f_{ai}{}^j(z_A) W_j, \\ \{W_i, W_j\} &= f_{ij}{}^k(z_A) W_k, \end{aligned} \quad (5)$$

where  $f_{ab}{}^c$  is independent of  $z_A$ . In particular, we note that both the  $T_a$  and the  $W_i$  form subalgebras of the constraint algebra. Corresponding to the split of the generators, it is natural to also split the ghosts as  $c^\alpha = (c^a, \gamma^i)$  and  $b_\alpha = (b_a, \beta_i)$ . The following theorem can now be proved.

*Theorem.* The BRST charge  $Q_B$  corresponding to the constraint algebra (5) can be written in the form

$$Q_B = Q_0 + Q_1, \quad \{Q_0, Q_0\} = \{Q_1, Q_1\} = \{Q_0, Q_1\} = 0, \quad (6)$$

with

$$\begin{aligned} Q_0 &= c^a T_a - \frac{1}{2} f_{ab}{}^c c^b c^a b_c - f_{ai}{}^j \gamma^i c^a \beta_j, \\ Q_1 &= \gamma^i W_i - \frac{1}{2} f_{ij}{}^k \gamma^j \gamma^i \beta_k. \end{aligned} \quad (7)$$

The proof of this theorem is rather simple and relies on the fact that, due to the particular form of the constraint algebra (5), the Jacobi identity (4) decomposes into three independent sets of identities,

$$f_{[ab}{}^e f_{c]e}{}^d = 0,$$

$$f_{ab}{}^d f_{cd}{}^j - 2f_{ai}{}^k f_{bk}{}^j + 2\{f_{ai}{}^j, T_b\} = 0, \quad (8)$$

$$f_{[ij}{}^l f_{kl]}{}^m - \{f_{[ij}{}^m, W_{k]}{}^l\} = 0, \quad (9)$$

$$2f_{ai}{}^k f_{jk}{}^l + f_{ij}{}^k f_{ak}{}^l - 2\{f_{ai}{}^l, W_j\} - \{f_{ij}{}^l, T_a\} = 0, \quad (10)$$

which can separately be used to show that  $\{Q_0, Q_0\} = 0$ ,  $\{Q_1, Q_1\} = 0$  and  $\{Q_0, Q_1\} = 0$ , respectively.

### 3. Examples

(1) As a first example we discuss the classical  $w_3$ -algebra which has a spin-2 generator  $T(z)$  and a spin-3 generator  $W(z)$ . In conformal OPE language the algebra takes the following form<sup>#2</sup>:

$$\begin{aligned} T(z)T(w) &= \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \dots, \\ T(z)W(w) &= \frac{3W(w)}{(z-w)^2} + \frac{\partial W(w)}{z-w} + \dots, \\ W(z)W(w) &= \frac{2TT}{3(z-w)^2} + \frac{\partial(TT)}{3(z-w)} + \dots \end{aligned} \quad (11)$$

A multi-scalar realisation of the algebra is given by [16]

$$\begin{aligned} T &= -\frac{1}{2}(\partial\phi_1)^2 + T_X, \\ W &= \frac{i}{3\sqrt{6}}\{(\partial\phi_1)^3 + 6(\partial\phi_1)T_X\}. \end{aligned} \quad (12)$$

where  $T_X$  is a multi-scalar realisation of the Virasoro algebra. Note that the  $w_3$ -algebra (11) is not yet of the form (5) and therefore we cannot apply our theorem. To bring the algebra in the desired form we make the following redefinition of the generators:

$$\begin{aligned} \tilde{T} &= T, \\ \tilde{W} &= W - \frac{2i}{\sqrt{6}}(\partial\phi_1)T = \frac{4i}{3\sqrt{6}}(\partial\phi_1)^3. \end{aligned} \quad (13)$$

The algebra then becomes

$$\begin{aligned} \tilde{T}(z)\tilde{T}(w) &= \frac{2\tilde{T}(w)}{(z-w)^2} + \frac{\partial\tilde{T}(w)}{z-w} + \dots, \\ \tilde{T}(z)\tilde{W}(w) &= \frac{3\tilde{W}(w)}{(z-w)^2} + \frac{\partial\tilde{W}(w)}{z-w} + \dots, \\ \tilde{W}(z)\tilde{W}(w) &= \frac{-2i\sqrt{6}\partial\phi_1\tilde{W}}{(z-w)^2} + \frac{-i\sqrt{6}\partial(\partial\phi_1\tilde{W})}{z-w} + \dots, \end{aligned} \quad (14)$$

<sup>#2</sup> To facilitate the transition to the quantum case we represent from now on Poisson brackets by Operator Product Expansions where it is understood that at the l.h.s. of the equation only *single* contractions between the fundamental fields are taken. Later on, when we consider the quantum case, it is understood that *multiple* contractions are taken into account as well.

which is indeed of the form (5). Applying our theorem, we can now write the BRST charge  $Q_B = \oint (dz/2\pi i) J_B$  in the form (6), (7) with the BRST currents  $J_0$  and  $J_1$  corresponding to  $Q_0$  and  $Q_1$ , respectively, given by

$$\begin{aligned} J_0 &= c\{\tilde{T} + T_{(\gamma,\beta)} + \frac{1}{2}T_{(c,b)}\}, \\ J_1 &= \gamma\{\tilde{W} - i\sqrt{6}\partial\phi_1\partial\gamma\beta\}. \end{aligned} \quad (15)$$

Here  $(c, b)$  and  $(\gamma, \beta)$  are the usual spin-2 and spin-3 ghost fields whose energy-momentum tensors are given by

$$T_{(c,b)} = -2b\partial c - \partial bc, \quad (16)$$

$$T_{(\gamma,\beta)} = -3\beta\partial\gamma - 2\partial\beta\gamma. \quad (17)$$

(2) As a second example we consider the modified  $w_3$ -algebra of [7,15]. Let  $\{T_M, W_M\}$  and  $\{T_L, W_L\}$  be two commuting copies of the classical  $w_3$ -algebra. For the matter sector we again use the realisation (12)<sup>#3</sup>.

The matter and Liouville parts can be combined into a single modified  $w_3$ -algebra by defining the new generators  $T = T_M + T_L$  and  $W = W_M + iW_L$  [7,15]. This algebra then takes the form of a modified  $w_3$ -algebra where, instead of the third line of (11), we have

$$W(z)W(w) = \frac{2(T_M - T_L)T}{3(z-w)^2} + \frac{\partial\{(T_M - T_L)T\}}{3(z-w)} + \dots \quad (18)$$

Note that  $T_M - T_L$  appears as a structure function, i.e. the algebra is of the so-called soft type.

Again, a redefinition of the generators can be made such that the algebra is of the form (5). In the present case, this redefinition is given by

$$\begin{aligned} \tilde{T} &= T, \\ \tilde{W} &= W - \frac{2i}{\sqrt{6}}(\partial\phi_1)T = \frac{4i}{3\sqrt{6}}(\partial\phi_1)^3 - \frac{2i}{\sqrt{6}}\partial\phi_1 T_L + iW_L \end{aligned} \quad (19)$$

In terms of the new generators the algebra is given by the same formula (14) which we already encountered in the case of the unmodified  $w_3$ -algebra. The same applies to the BRST charge. It is again given by eq. (15).

As a byproduct of our manipulations, we see that, by allowing realization-dependent redefinitions of the generators, the classical  $w_3$ -algebra and the modified  $w_3$ -algebra can be brought into exactly the same form as given in (14). Of course, the *realization* of the generators is different in both cases. Obviously, the previous example is contained in this example as can be seen by putting  $T_L = W_L = 0$ .

#### 4. Quantisation

To describe the quantisation, it is convenient to use the Batalin-Vilkovisky (BV) formalism [17]. This formalism enables one, under the condition that the theory has no anomalies, to derive from a given classical BRST charge a nilpotent BRST operator by means of an iterative procedure. This BV formalism has for instance been applied in [15] to derive the quantum BRST operator corresponding to the modified  $w_3$ -algebra. A priori, it is not guaranteed by this procedure that a classical BRST charge with the property (6) carries over into a quantum

<sup>#3</sup> Note that, instead of using realisation (12) of the matter sector, we could also use such a realisation for the Liouville sector. One could then perform the same analysis as given below leading to the same results with everywhere matter replaced by Liouville.

BRST operator with the property (1). However, in all cases we have investigated, the property (1) turns out to hold.

The expression for the quantum BRST operator corresponding to the redefined  $w_3$ -algebra can be found in [4] so we will not repeat it here. In [4] it was noted that the quantum BRST operator corresponding to the redefined algebra (14) can be related to the quantum BRST operator corresponding to the unmodified  $w_3$ -algebra by means of a field redefinition of the ghosts and the matter fields. In case of the redefined modified  $w_3$ -algebra, instead of deriving the quantum BRST operator through the BV formalism, we found it much simpler to use the quantum BRST operator corresponding to the modified  $w_3$ -algebra given in [7] and to apply a similar redefinition on it. Of course, the result is the same as what would follow from the BV formalism.

We have found that the quantum counterparts of the classical BRST currents (15) are given by

$$J_0 = c \{ T_{\phi_1} + T_X + T_L + T_{(\gamma,\beta)} + \frac{1}{2} T_{(c,b)} \}, \quad (20)$$

$$J_1 = \gamma \left[ \frac{i}{3\sqrt{6}} \{ 4(\partial\phi_1)^3 - 12q_1\partial\phi_1\partial^2\phi_1 + (-15 + 4q_1^2)\partial^3\phi_1 \} + i \left\{ W_L - \frac{2}{\sqrt{6}}\partial\phi_1 T_L + \frac{1}{\sqrt{6}}q_1\partial T_L \right\} \right. \\ \left. - i\sqrt{6} \{ \partial\phi_1\partial\gamma\beta + \frac{1}{3}q_1\partial\beta\partial\gamma \} \right]. \quad (21)$$

The quantum energy-momentum tensor for the  $\phi_1$  scalar is given by

$$T_{\phi_1} = -\frac{1}{2}(\partial\phi_1)^2 + q_1\partial^2\phi_1 \quad (22)$$

This scalar contributes  $c_{\phi_1} = 1 + 12q_1^2$  to the total central charge, while the additional matter scalars, with energy-momentum tensor  $T_X$ , contribute  $c_X = 1 + 4q_1^2$  [16]. The total central charge contribution of the matter and Liouville sectors is thus given by  $c_M = c_{\phi_1} + c_X = 2 + 16q_1^2$  and  $c_L$ , respectively. Furthermore, the energy-momentum tensors  $T_{(c,b)}$  and  $T_{(\gamma,\beta)}$  of the spin-2 and spin-3 ghosts satisfy a Virasoro algebra with central charge  $-26$  and  $-74$ , respectively. Finally, the generators  $(T_L, W_L)$  satisfy a quantum  $W_3$ -algebra with central charge  $c_L$ . The nilpotency of the BRST operator requires that  $c_M + c_L = 100$ .

## 5. Comments

It has been suspected for a long time that various non-critical  $W_n$  strings are somehow related. Evidence supporting this can be found in the appearance of  $W_n$  constraints in the double scaling limit of multimatrix models and in the results for the spectrum of pure  $W_3$ -gravity [18,19]. We believe that the present work strongly suggests a larger scheme which would enable one to establish relations across models.

In order to formulate our conjecture, we make two observations. In [18,20], it was shown that a set of  $W_{n-1}$  currents and one free scalar field with a fixed background charge allows one to construct the  $W_n$  currents. A second point is that the work in [7,15] very probably generalizes to arbitrary  $W_n$  models in the following way. Consider two mutually commuting copies of the  $W_n$  algebra, generated by  $T_M^{(j)}$  and  $T_L^{(j)}$  resp., where  $2 \leq j \leq n$ . The total currents  $T^{(j)}$  are  $T^{(j)} \equiv T_M^{(j)} + i^{j-2} T_L^{(j)}$ <sup>#4</sup>. We next write the Liouville currents in terms of a scalar field  $\phi$  and  $W_{n-1}$  currents<sup>#5</sup>. We conjecture that the BRST operator for this system can be written in the form (1) where  $Q_0$  is the BRST operator for a  $W_{n-1}$  system. In  $Q_0$ , the ghosts for the spin  $n$  symmetry together with the scalar  $\phi$  have become “matter” for the  $W_{n-1}$  system. This conjecture can presumably be proven by combining the two observations above.

<sup>#4</sup> This observation was made in collaboration with X. Shen.

<sup>#5</sup> This is an alternative to what we did in example 2 of section 3 where we specified a realization of the matter sector. See also the footnote in that section

The following counting argument supports our conjecture. Consider  $W_n$  gravity coupled to  $(p, q)$   $W_n$  minimal matter, i.e. non-critical  $W_n$  strings. The central charge  $c_M$  of the  $(p, q)$  minimal model is given by

$$c_M = (n-1)\{1 - n(n+1)Q_M^2\}, \quad (23)$$

where

$$Q_M = \sqrt{\frac{p}{q}} - \sqrt{\frac{q}{p}}. \quad (24)$$

The  $W_n$  gravity sector is an  $Sl(n, \mathbb{R})$  Toda system with central charge  $c_T$

$$c_T = (n-1)\{1 + n(n+1)Q_T^2\}, \quad (25)$$

where

$$Q_T = \sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}}. \quad (26)$$

The two central charges add up to

$$c_M + c_T = 2(n-1)(2n^2 + 2n + 1), \quad (27)$$

which gets precisely cancelled by the  $W_n$  ghost contributions to the total  $c$ . Following [18,20], the  $W_n$  Toda currents can now be rewritten in terms of  $W_{n-1}$  currents and one scalar field  $\phi$ , with central charge  $c_\phi = 1 + 3n(n-1)Q_T^2$ . If our conjecture is true, the  $W_n$  minimal matter, together with  $\phi$  and the ghost and the anti-ghost of the spin  $n$  symmetry (which has a central charge  $c_{(c(n), b(n))} = -2(6n^2 - 6n + 1)$ ), combine to a matter sector with  $W_{n-1}$  symmetry. This matter has central charge

$$\tilde{c}_M = c_M + c_\phi + c_{(c(n), b(n))} = (n-2)\{1 - n(n-1)Q_M^2\}, \quad (28)$$

which is precisely what we expect for the  $(p, q)$   $W_{n-1}$  minimal model. Our conjecture suggests novel realizations of  $(p, q)$   $W_n$  minimal models in terms of  $n+k-1$  scalar fields and  $k$   $b-c$  systems with  $1 \leq k \leq n-2$ .

If we restrict ourselves to unitary minimal models, i.e. choosing  $q+1=p$ , we get that the following relation across lines should exist:

pure  $W_2$   
gravity

Ising ( $c = 1/2$ )  $\leftrightarrow$  pure  $W_3$   
+  $W_2$  gravity gravity

tricritical  $\leftrightarrow$  3-state  $\leftrightarrow$  pure  $W_4$   
Ising ( $c = 7/10$ ) Potts ( $c = 4/5$ ) gravity  
+  $W_2$  gravity +  $W_3$  gravity

$\vdots$   $\vdots$   $\vdots$   $\ddots$

(29)

Our conjecture amounts to the claim that the spectrum of every model in the table contains as a subsector the spectrum of the model to its left. In other words, a  $(p, q)$  non-critical  $W_n$  string contains in its spectrum the spectrum of  $(p, q)$  non-critical  $W_{n-k}$  strings where  $1 \leq k \leq n-2$ .

Repeated applications of our conjecture would result in the factorization of  $Q$  into a Virasoro BRST charge and  $n-2$  other, mutually anticommuting, nilpotent charges. This is supported by the fact that any classical  $w_n$ -algebra can be realised in terms of an arbitrary number of scalars  $X^\mu$  and  $n-2$  special scalars  $\phi_i$  ( $i = 1, 2, \dots, n-2$ ) as follows:

$$\begin{aligned}
 T &= T_X + \sum_{i=1}^{n-2} T_{\phi_i}, \\
 &\vdots \\
 W^{(n-3)} &\sim \sum_h (\partial \phi_{(n-3)})^{n-1-h} (\partial \phi_{(n-2)})^h, \\
 W^{(n-2)} &\sim (\partial \phi_{(n-2)})^n.
 \end{aligned} \tag{30}$$

The generic structure of a  $w_n$ -algebra in this basis is of the form

$$\begin{aligned}
 [T, T] &\rightarrow T, \\
 &\vdots \\
 [W^{(n-3)}, W^{(n-3)}] &\rightarrow W^{(n-3)}, W^{(n-2)}, \\
 [W^{(n-3)}, W^{(n-2)}] &\rightarrow W^{(n-3)}, W^{(n-2)}, \\
 [W^{(n-2)}, W^{(n-2)}] &\rightarrow W^{(n-2)}.
 \end{aligned} \tag{31}$$

This generalizes the structure given in (5). It would be interesting to see whether an algebra of the type (31) allows a multi-split of the corresponding BRST charge of the form

$$Q_B = \sum_{i=0}^{n-2} Q_i, \quad \{Q_i, Q_j\} = 0. \tag{32}$$

One could even consider taking the limit  $n \rightarrow \infty$  and construct a  $W_\infty$ -string theory. More details on the generic case of classical  $w_n$  algebras and their quantisation will be given elsewhere [21].

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